The Dynamic Response of Multidirectional Functionally Graded Plates Impacted by Blast Loading

Terry Hause[a], Ph.D.

[a]Research Mechanical Engineer, U.S. Army RDECOM-TARDEC, Warren, MI 48397

ABSTRACT

The theoretical model for the dynamic response of multi-directional functionally graded thin plates under an in-air blast loading from a Friedlander type pressure loading is presented. The theory is presented in the context of the classical linear plate theory (CPT) which is based on the Kirchoff-Love assumptions. The plate is assumed to be thin, in-plane strains are small compared to unity, and the transverse and normal strains are considered to be negligible. Additionally, the theoretical model assumes that the material properties of the two constituent materials vary in all three coordinate directions. This implies in-plane as well as through the thickness grading according to 3 independent power law distributions. Simply supported boundary conditions are assumed along all four edges. The governing equations of motion are derived through the use of Hamilton's Principle. The dynamic response is determined through the use of numerical integration, using the Gaussian-Quadrature Method, the Galerkin Method, and the Fourth-Order Runge-Kutta Method with zero initial conditions. Results are presented using the technique of spatial tailoring to determine the optimization of the 3D-Grading from a response standpoint. Finally, validations are made with simpler cases found within the literature.

Key Words: Functionally Graded; Dynamic Response; Blast; Transient Response; Multi-directional; Plates

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1. Introduction

With the demand for protection from the effects of war, combat military vehicles need to be fully armored to protect the soldier from such things as gun fire (and or penetration), land mines, IED's, Fire, etc., One of the most daunting tasks is to engineer a military vehicle to withstand blast from an IED or land mine. The most important areas of the vehicle are the hull which resides underneath the vehicle and armor plating surrounding various key locations of the vehicle to protect the occupants inside. In the case of the hull, a structurally sound hull would protect the occupants from large crushing floor accelerations impacting the lower part of the body. For this reason, such new materials as multidirectional functionally graded materials should be explored to determine any possible advantages over other types of materials.

Multidirectional materials are materials which exhibit properties which are a function of all three coordinate directions as opposed to just the transverse direction found in conventional functionally graded materials. This leads to the concept of spatial tailoring whereby by one can manipulate the distribution of the constituent materials in all three coordinate directions to achieve an optimum distribution which will provide the desired structural response. This is the focus of this paper.

Currently, there is a disparity of research and information on this topic. Very little exists in the literature. Due to the complexity of the theoretical development, considering a thorough treatment of the theoretical model, which most likely, would require advanced numerical solution procedures, a simplified model based on the linearized theory is adopted to serve as a basic foundation upon which to build upon. Three independent power law distributions are presented which describe mathematically the grading in all three coordinate directions. The

transverse or through-the-thickness grading as well as the in-plane grading is assumed to be symmetric on both the top and bottom faces.

The present linear theory includes damping effects, transverse inertia, and a transient normal loading, due to a Friedlander-type pressure-time impact.

It should be emphasized that this theoretical model is only valid within the elastic region of the material. To remain in the elastic region, the magnitude of the transverse pressure acting on the plate would most likely need to be of a low to moderate intensity. The stresses and strains would have to be evaluated to determine if they reside beyond the elastic limit.

2. Basic Assumptions and Preliminaries

Shown, in Figure 1, is a pictorial representation of a multidirectional functionally graded plate referred to an Orthogonal Cartesian Coordinate System (x, y, z), where z is measured positive in the upwards direction from the mid-surface of the plate. While, h is the uniform thickness of the plate. Let any two constituent materials comprise a functionally graded plate. Then by applying the rule of mixtures, a generic property $P(x, y, z)$ can be expressed as

$$P(x, y, z) = P_1 V_1(x, y, z) + P_2 V_2(x, y, z) \tag{1}$$

Where P represents the Young's Modulus, Density, Poisson's Ratio, Coefficient of Thermal Expansion, Etc., while, V_1, V_2 represent the volume fractions of the two constituent materials which must obey the following relationship

$$V_1(x, y, z) + V_2(x, y, z) = 1\,. \tag{2}$$

With the use of Eq. (2), Eq. (1) can be expressed as

$$P(x, y, z) = [P_1 - P_2]V_1(x, y, z) + P_2\,. \tag{3}$$

Expressing Eq. (3) in variable separable form gives

$$P(x,y,z) = [P_1 - P_2]V_{cx}(x)V_{cy}(y)V_{cz}(z) + P_2\,, \tag{4}$$

Where the chosen functional relationships for the volume fractions, $V_{cx}(x)$, $V_{cy}(y)$, and $V_{cz}(z)$ are given in a polynomial and power form as

$$V_{cx}(x) = \left[\frac{x}{L_1}\left(1 - \frac{x}{L_1}\right)\right]^{N_1}, \tag{5a}$$

$$V_{cy}(y) = \left[\frac{y}{L_2}\left(1 - \frac{y}{L_2}\right)\right]^{N_2}, \tag{5b}$$

$$V_{cz}(z) = \left(\frac{z}{h/2}\right)^M \left(\frac{1+\mathrm{sgn}(z)}{2}\right) + \left(\frac{-z}{h/2}\right)^M \left(\frac{1-\mathrm{sgn}(z)}{2}\right) \tag{5c}$$

The Signum function is defined as

$$\mathrm{sgn}(z) = \begin{cases} 1, & z > 0 \\ 0, & z = 0 \\ -1 & z < 0 \end{cases} \tag{6}$$

N_1, N_2, and M are referred to as the volume fraction indexes providing a measure of the variation of the material profile through the structure in all three coordinate directions. This chosen grading of the constituent materials throughout the plate leads to a symmetric distribution in all three coordinate directions. It should be noted, depending on the 3D grading desired that other possible functional relationships for the volume fraction could be utilized.

The chosen constituent materials for this paper are ceramic and metal. This leads to the expression of the material properties given by

$$[E(x,y,z),\ \rho(x,y,z)] = [E_{cm}\,,\ \rho_{cm}]V_1(x,y,z) + [E_m\,,\ \rho_m] \tag{7}$$

Where,

$$E_{cm} = E_c - E_m, \qquad \rho_{cm} = \rho_c - \rho_m . \tag{8}$$

It should be noted that the variation of Poisson's ratio, $\nu(x, y, z)$, is approximated as being constant throughout the material grading of the structure with the assumption that the effect of any point-wise variation on the dynamic structure response would be minimal and or negligible for a thin plate.

3. Kinematic Equations

3.1 The Displacement Field

Based on the classical plate theory, the 3-D displacement relationships are expressed as

$$u = u_0 - z\frac{\partial w_0}{\partial x} \tag{9a}$$

$$\upsilon = \upsilon_0 - z\frac{\partial w_0}{\partial y} \tag{9b}$$

$$w = w_0 \tag{9c}$$

Where, (u, υ, w) are the 3-D displacement quantities and (u_0, υ_0, w_0) are 2-D displacement quantities of the mid-surface of the plate.

3.2 Linear Strain-Displacement Relationships

Assuming the transverse and normal strains are negligible, The linear strain displacement relationships for plane stress are given by

$$e_{xx} = \frac{\partial u}{\partial x} \tag{10a}$$

$$e_{yy} = \frac{\partial \upsilon}{\partial y} \tag{10b}$$

$$\gamma_{xy} = 2e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x} \tag{10c}$$

$$\gamma_{xz} = \gamma_{yz} = e_{zz} = 0 \tag{10d}$$

With the use of the displacement relationships, Eqs. (9a-c), the strain-displacement relationships, Eqs. (10a-c,) can be expressed in terms of 2-D displacement quantities as

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \tag{11}$$

Where

$$\varepsilon_{xx}^0 = \frac{\partial u_0}{\partial x}, \quad \varepsilon_{yy}^0 = \frac{\partial \upsilon_0}{\partial y}, \quad \gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial \upsilon_0}{\partial x} \tag{12a}$$

$$\kappa_{xx} = -\frac{\partial^2 w_0}{\partial x^2}, \quad \kappa_{yy} = -\frac{\partial^2 w_0}{\partial y^2}, \quad \kappa_{xy} = -2\frac{\partial^2 w_0}{\partial x \partial y} \tag{12b}$$

In the above expressions, $\left(\varepsilon_{xx}^0, \varepsilon_{yy}^0, \gamma_{xy}^0\right)$ are known as the membrane strains; While, $(\kappa_{xx}, \kappa_{yy}, \kappa_{xy})$ are the bending strains.

4. Constitutive Equations

The constitutive equations for a point-wise isotropic material are given by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11}(x,y,z) & Q_{12}(x,y,z) & 0 \\ Q_{12}(x,y,z) & Q_{22}(x,y,z) & 0 \\ 0 & 0 & Q_{66}(x,y,z) \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{Bmatrix} \tag{13}$$

$$\tau_{xz} = \tau_{yz} = \sigma_{zz} = 0$$

Where the material stiffnesses $Q_{ij}(x,y,z)$, (i, j=1,2,6) are given by

$$Q_{11} = Q_{22} = \frac{E(x,y,z)}{1-\nu^2}, Q_{12} = \frac{\nu E(x,y,z)}{1-\nu^2}, Q_{66} = \frac{E(x,y,z)}{2(1+\nu)}, \; Q_{16} = Q_{26} = 0 \tag{14}$$

5. Equations of Motion

Adopting an energy approach, the equations of motion are derived through the use of Hamilton's Principle. It is provided as

$$\delta J = \delta \int_{t_0}^{t_1} (T - U + V) dt = 0 \tag{15}$$

Where t_0, t_1 are two arbitrary instants in time. *U* denotes the strain energy, *V* denotes the work done by surface tractions, edge loads, body forces, and damping forces. For this paper, there are only surface tractions in the form of a transient transversal loading and damping forces. *T* denotes the kinetic energy of the structure, while δ is the variational operator. The strain energy, *U* is given by

$$\delta U = \int_{\Omega} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{xx} \delta e_{xx} + \sigma_{yy} \delta e_{yy} + \tau_{xy} \gamma_{xy}) dz d\Omega \tag{16}$$

Ω denotes the mid-surface area of the plate. The work done by the transverse pressure and damping forces is given by

$$\delta V = \int_{\Omega} [P_t(x, y, t) - \mu(x, y) \dot{w}_0)] \delta w_0 d\Omega \tag{17}$$

In the above expression, P_t is the distributed force at the top surface, and $\mu(x, y)$ is the damping coefficient per unit area of the plate. It should also be noted that although $\mu = \mu(x, y, z)$, the damping coefficient is approximated as being constant through the thickness of the thin plate but can vary across the 2D plane of the plate. The transverse kinetic energy is given by

$$\delta T = \int_0^t \int_\Omega \rho(x,y)\ddot{w}_0 \delta w_0 \, d\Omega \, dt \tag{18}$$

Where, $\rho(x,y)$, the inertia term is given by

$$\rho(x,y) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(x,y,z)dz \tag{19}$$

where $\rho(x,y,z)$ is the mass per unit volume.

With the expressions for the strain energy, Eq. (16), the work, Eq. (17), and the kinetic energy Eq. (18), in hand and considering Eqs. (11), (12a,b), (15), carrying out the integration throughout the thickness, integrating by parts where ever feasible, and taking into consideration the arbitrary and independent character of variations results in the equations of motion and the associated boundary conditions which are provided in terms of stress resultants and stress couples as

$$\delta u_0 : \quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{20}$$

$$\delta \upsilon_0 : \quad \frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \tag{21}$$

$$\delta w_0 : \quad \frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + P_t(x,y,t) - \rho(x,y)\ddot{w}_0 - \mu(x,y)\dot{w}_0 = 0\,. \tag{22}$$

The associated boundary conditions become:

Along the edges $x = (0, L_1)$

$$N_{xx} = 0 \quad \text{or} \quad \delta u_0 = 0 \Rightarrow u_0 = \tilde{u}_0 \tag{23}$$

$$N_{xy} = 0 \quad \text{or} \quad \delta \upsilon_0 = 0 \Rightarrow \upsilon_0 = \tilde{\upsilon}_0 \tag{24}$$

$$M_{xx} = 0 \quad \text{or} \quad \delta\left(\frac{\partial w_0}{\partial x}\right) = 0 \Rightarrow \frac{\partial w_0}{\partial x} = \frac{\partial \tilde{w}_0}{\partial x} \tag{25}$$

$$\frac{\partial M_{xx}}{\partial x} + 2\frac{\partial M_{xy}}{\partial y} = 0 \qquad \text{or} \qquad \delta w_0 = 0 \Rightarrow w_0 = \tilde{w}_0 \tag{26}$$

Along the edges $y = (0, L_2)$

$$N_{xy} = 0 \qquad \text{or} \qquad \delta u_0 = 0 \Rightarrow u_0 = \tilde{u}_0 \tag{27}$$

$$N_{yy} = 0 \qquad \text{or} \qquad \delta \upsilon_0 = 0 \Rightarrow \upsilon_0 = \tilde{\upsilon}_0 \tag{28}$$

$$M_{yy} = 0 \qquad \text{or} \qquad \delta\left(\frac{\partial w_0}{\partial y}\right) = 0 \Rightarrow \frac{\partial w_0}{\partial y} = \frac{\partial \tilde{w}_0}{\partial y} \tag{29}$$

$$\frac{\partial M_{yy}}{\partial y} + 2\frac{\partial M_{xy}}{\partial x} = 0 \qquad \text{or} \qquad \delta w_0 = 0 \Rightarrow w_0 = \tilde{w}_0 \tag{30}$$

The quantities with over-carets are prescribed quantities along the boundary of the plate. It is desired to express the governing equation of motion and the associated boundary conditions in terms of displacements. To achieve this end, the stress resultants and stress couples are defined and expressed in terms of displacements as presented below.

The stress resultants and stress couple resultants are defined as

$$\left(N_{\alpha\beta}, M_{\alpha\beta}\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta}(1, z)\,dz \quad , \tag{31}$$

where $\alpha, \beta = (x, y)$. Substitution of the constitutive Eqs. (13) and the strain-displacement relations Eqs. (11), results in a relationship between the stress resultants and stress couples in terms of the mid-surface and bending strain components expressed as

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} (\begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \tag{32}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \tag{33}$$

Where, the global stiffness quantities are defined as

$$[A_{ij}(x,y), B_{ij}(x,y), D_{ij}(x,y)] = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}(x,y,z)(1,z,z^2)dz, \qquad (i,\ j = 1,\ 2,\ 6) \tag{34}$$

It should be noted that in the case of symmetric grading in all three coordinate directions $[B_{ij}] = 0$. As a result, these quantities are excluded from Eqs. (32) and (33). With the use of Eqs. (7) and (14), and carrying out the indicated integration within Eq. (34), the global stiffness quantities can be expressed as

$$(A_{11}, B_{11}, D_{11}) = \frac{1}{1-\nu^2}(E_1, E_2, E_3) \tag{35a}$$

$$(A_{12}, B_{12}, D_{12}) = \frac{\nu}{1-\nu^2}(E_1, E_2, E_3) \tag{35b}$$

$$(A_{66}, B_{66}, D_{66}) = \frac{1}{2(1+\nu)}(E_1, E_2, E_3) \tag{35c}$$

Where,

$$E_1(x,y) = \frac{E_{cm}h}{M+1}V_{cx}(x)V_{cy}(y) + hE_m, \quad E_2(x,y) = 0 \tag{36a,b}$$

$$E_3(x,y) = \frac{E_{cm}h^3}{4(M+3)}V_{cx}(x)V_{cy}(y) + \frac{E_m h^3}{12} \tag{36c}$$

Making use of Eqs. (12a,b), (32), (33) with Eqs. (20)-(22) and adopting some basic algebraic techniques, yields the equations of motion and boundary terms, in terms of displacements as,

$$\delta u_0: \quad A_{11}(x,y)\frac{\partial^2 u_0}{\partial x^2} + [A_{12}(x,y) + A_{66}(x,y)]\frac{\partial^2 \upsilon_0}{\partial x \partial y} + A_{66}(x,y)\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial A_{11}(x,y)}{\partial x}\frac{\partial u_0}{\partial x} + \frac{\partial A_{66}(x,y)}{\partial y}\left(\frac{\partial u_0}{\partial y} + \frac{\partial \upsilon_0}{\partial x}\right) + \frac{\partial A_{12}(x,y)}{\partial x}\frac{\partial \upsilon_0}{\partial y} = 0 \tag{37}$$

$$\delta \upsilon_0: \quad A_{22}(x,y)\frac{\partial^2 \upsilon_0}{\partial y^2} + [A_{12}(x,y) + A_{66}(x,y)]\frac{\partial^2 u_0}{\partial x \partial y} + A_{66}(x,y)\frac{\partial^2 \upsilon_0}{\partial x^2} + \frac{\partial A_{22}(x,y)}{\partial y}\frac{\partial \upsilon_0}{\partial y} + \frac{\partial A_{66}(x,y)}{\partial x}\left(\frac{\partial u_0}{\partial y} + \frac{\partial \upsilon_0}{\partial x}\right) + \frac{\partial A_{12}(x,y)}{\partial y}\frac{\partial u_0}{\partial x} = 0 \tag{38}$$

$$\delta w_0: \quad D_{11}(x,y)\nabla^4 w_0 + 2\frac{\partial D_{11}(x,y)}{\partial x}\frac{\partial(\nabla^2 w_0)}{\partial x} + 2\frac{\partial D_{11}(x,y)}{\partial y}\frac{\partial(\nabla^2 w_0)}{\partial y} + \nabla^2 D_{11}(x,y)(\nabla^2 w_0) - (1-\nu)\left(\frac{\partial^2 D_{11}(x,y)}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} - 2\frac{\partial^2 D_{11}(x,y)}{\partial x \partial y}\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial^2 D_{11}(x,y)}{\partial y^2}\frac{\partial^2 w_0}{\partial x^2}\right) + \rho(x,y)\ddot{w}_0 + \mu(x,y)\dot{w}_0 = P_t(x,y,t) \tag{39}$$

with the associated boundary terms as,

Along the edges $x = (0, L_1)$

$$A_{11}(x,y)\frac{\partial u_0}{\partial x} + A_{12}(x,y)\frac{\partial \upsilon_0}{\partial y} = 0 \quad \text{or} \quad u_0 = \tilde{u}_0 \tag{40}$$

$$A_{66}(x,y)\left(\frac{\partial u_0}{\partial y} + \frac{\partial \upsilon_0}{\partial x}\right) = 0 \quad \text{or} \quad \upsilon_0 = \tilde{\upsilon}_0 \tag{41}$$

$$-D_{11}(x,y)\frac{\partial^2 w_0}{\partial x^2} - D_{12}(x,y)\frac{\partial^2 w_0}{\partial y^2} = 0 \quad \text{or} \quad \frac{\partial w_0}{\partial x} = \frac{\partial \tilde{w}_0}{\partial x} \tag{42}$$

$$-D_{11}(x,y)\frac{\partial^3 w_0}{\partial x^3}-[D_{12}(x,y)+4D_{66}(x,y)]\frac{\partial^3 w_0}{\partial x\partial y^2}-\frac{\partial D_{11}(x,y)}{\partial x}\frac{\partial^2 w_0}{\partial x^2}-4\frac{\partial D_{66}(x,y)}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y}-\frac{\partial D_{12}(x,y)}{\partial x}\frac{\partial^2 w_0}{\partial y^2}=0 \quad \text{or} \quad w=\tilde{w}_0 \tag{43}$$

Along the edges $y=(0,L_2)$

$$A_{66}(x,y)\left(\frac{\partial u_0}{\partial y}+\frac{\partial \upsilon_0}{\partial x}\right)=0 \quad \text{or} \quad u_0=\tilde{u}_0 \tag{44}$$

$$A_{22}(x,y)\frac{\partial \upsilon_0}{\partial y}+A_{12}(x,y)\frac{\partial u_0}{\partial x}=0 \quad \text{or} \quad \upsilon_0=\tilde{\upsilon}_0 \tag{45}$$

$$-D_{22}(x,y)\frac{\partial^2 w_0}{\partial y^2}-D_{12}(x,y)\frac{\partial^2 w_0}{\partial x^2}=0 \quad \text{or} \quad \frac{\partial w_0}{\partial y}=\frac{\partial \tilde{w}_0}{\partial y} \tag{46}$$

$$-D_{22}(x,y)\frac{\partial^3 w_0}{\partial y^3}-[D_{12}(x,y)+4D_{66}(x,y)]\frac{\partial^3 w_0}{\partial x^2\partial y}-\frac{\partial D_{22}(x,y)}{\partial y}\frac{\partial^2 w_0}{\partial y^2}-4\frac{\partial D_{66}(x,y)}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}-\frac{\partial D_{12}(x,y)}{\partial y}\frac{\partial^2 w_0}{\partial x^2}=0 \quad \text{or} \quad w_0=\tilde{w}_0 \tag{47}$$

. By observation, it is seen that the first two equations of motion, Eqs. (20) and (21) governing the in-plane motion are decoupled from the third equation of motion, Eq. (22), which governs the transverse or bending motion of the plate. As a result of the decoupling, the governing system (The equations of motion and boundary terms) is reduced to one governing equation of motion and two boundary conditions along each edge of the plate (Note: a fourth-order differential equation requires two boundary conditions along each edge). This reduces the governing system to

$$\delta w_0 : \quad D_{11}(x,y)\nabla^4 w_0 + 2\frac{\partial D_{11}(x,y)}{\partial x}\frac{\partial(\nabla^2 w_0)}{\partial x} + 2\frac{\partial D_{11}(x,y)}{\partial y}\frac{\partial(\nabla^2 w_0)}{\partial y} + \nabla^2 D_{11}(x,y)(\nabla^2 w_0) - (1-\nu)\left(\frac{\partial^2 D_{11}(x,y)}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} - 2\frac{\partial^2 D_{11}(x,y)}{\partial x\partial y}\frac{\partial^2 w_0}{\partial x\partial y} + \frac{\partial^2 D_{11}(x,y)}{\partial y^2}\frac{\partial^2 w_0}{\partial x^2}\right) + \rho(x,y)\ddot{w}_0 + \mu(x,y)\dot{w}_0 = P_t(x,y,t) \tag{48}$$

Along the edges $x = (0, L_1)$

$$-D_{11}(x,y)\frac{\partial^2 w_0}{\partial x^2} - D_{12}(x,y)\frac{\partial^2 w_0}{\partial y^2} = 0 \quad \text{or} \quad \frac{\partial w_0}{\partial x} = \frac{\partial \tilde{w}_0}{\partial x} \tag{49}$$

$$-D_{11}(x,y)\frac{\partial^3 w_0}{\partial x^3} - [D_{12}(x,y) + 4D_{66}(x,y)]\frac{\partial^3 w_0}{\partial x\partial y^2} - \frac{\partial D_{11}(x,y)}{\partial x}\frac{\partial^2 w_0}{\partial x^2} - 4\frac{\partial D_{66}(x,y)}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y} - \frac{\partial D_{12}(x,y)}{\partial x}\frac{\partial^2 w_0}{\partial y^2} = 0 \quad \text{or} \quad w = \tilde{w}_0 \tag{50}$$

Along the edges $y = (0, L_2)$

$$-D_{22}(x,y)\frac{\partial^2 w_0}{\partial y^2} - D_{12}(x,y)\frac{\partial^2 w_0}{\partial x^2} = 0 \quad \text{or} \quad \frac{\partial w_0}{\partial y} = \frac{\partial \tilde{w}_0}{\partial y} \tag{51}$$

$$-D_{22}(x,y)\frac{\partial^3 w_0}{\partial y^3} - [D_{12}(x,y) + 4D_{66}(x,y)]\frac{\partial^3 w_0}{\partial x^2\partial y} - \frac{\partial D_{22}(x,y)}{\partial y}\frac{\partial^2 w_0}{\partial y^2} - 4\frac{\partial D_{66}(x,y)}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y} - \frac{\partial D_{12}(x,y)}{\partial y}\frac{\partial^2 w_0}{\partial x^2} = 0 \quad \text{or} \quad w_0 = \tilde{w}_0 \tag{52}$$

For the problem at hand, *simply supported* boundary conditions are chosen. This leads to the following choice of boundary constraints expressed as:

Along the edges $x = (0, L_1)$

$$w_0 = 0, \qquad M_{xx} = -D_{11}(x,y)\frac{\partial^2 w_0}{\partial x^2} - D_{12}(x,y)\frac{\partial^2 w_0}{\partial y^2} = 0 \tag{53}$$

Along the edges $x = (0, L_2)$

$$w_0 = 0, \qquad M_{yy} = -D_{22}(x,y)\frac{\partial^2 w_0}{\partial y^2} - D_{12}(x,y)\frac{\partial^2 w_0}{\partial x^2} = 0\,. \tag{54}$$

6. Solution Methodology

To facilitate a solution of Eq. (48), the Galerkin Method is chosen which requires that both the essential (kinematic) and natural boundary conditions be fulfilled. To achieve this requirement, the transverse displacement, $w_0(x, y)$, is expressed in functional form as,

$$w_0(x,y,t) = W(t)\sin(\lambda_m x)\sin(\mu_n y) \tag{55}$$

Where $\lambda_m = m\pi/L_1$, $\mu_n = n\pi/L_2$ and (m ,n) are the number of sine half-waves in the corresponding directions. A suitable representation for $P_t(x, y, t)$ is given in terms of a Navier representation as

$$P_t(x,y,t) = P_{mn}(t)\sin(\lambda_m x)\sin(\mu_n y) \tag{56}$$

Integrating both sides of the above expression over the plate area gives

$$P_{mn}(t) = \frac{4}{L_1 L_2}\int_0^{L_2}\int_0^{L_1} P_t(x,y,t)\sin(\lambda_m x)\sin(\mu_n ydxdy \tag{57}$$

The pressure loading is given by the Friedlander expression as

$$P_t(x,y,t) = P_t(t) = (P_{S0} - P_0)\left[1 - \frac{t - t_a}{t_p}\right] e^{-(t - t_a)/t_p} \tag{58}$$

Substituting into Eq. (57) and integrating gives

$$P_{mn}(t) = 16\,P_t(t) \Big/ mn\,\pi^2 \tag{59}$$

In Eq. (58), P_{S0} is the peak overpressure above ambient pressure, P_0 is the ambient pressure, t_a is the time of arrival, t_p is the positive phase duration of the blast wave, t is the time, and α is known as the decay parameter which is determined by adjustment to a pressure curve from a blast test.

Applying the Galerkin Method to Eqs (48), with the use of Eqs (55), (56), and (59), gives

$$I_{mn}\ddot{W}_{mn}(t) + C_{mn}\dot{W}_{mn}(t) + K_{mn}W_{mn}(t) = \frac{16\,P_t(t)}{mn\,\pi^4}, \tag{60}$$

Where $W_{mn}(t)$ is the amplitude of deflection, I_{mn} is the plate inertia, C_{mn} is the overall damping coefficient of the plate, and K_{mn} is the overall stiffness of the plate all of which are given as

$$K_{mn} = \int_0^{L_2}\int_0^{L_1}\Bigg[\left(\lambda_m^2 + \mu_n^2\right)^2 D_{11}(x,y)s_{mx}^2 s_{ny}^2 - 2\lambda_m\left(\lambda_m^2 + \mu_n^2\right)^2 \frac{\partial D_{11}(x,y)}{\partial x} c_{mx} s_{mx} s_{ny}^2 - \tag{61}$$

$$- 2\mu_n\left(\lambda_m^2 + \mu_n^2\right)^2 \frac{\partial D_{11}(x,y)}{\partial y} s_{mx}^2 c_{ny} s_{ny} - \left(\lambda_m^2 + \nu\mu_n^2\right)\frac{\partial^2 D_{11}(x,y)}{\partial x^2} s_{mx}^2 s_{ny}^2 +$$

$$+ 2\lambda_m\mu_n(1-\nu)\frac{\partial^2 D_{11}(x,y)}{\partial x\partial y} c_{mx} s_{mx} c_{ny} s_{ny} - \left(\mu_n^2 + \nu\lambda_m^2\right)\frac{\partial^2 D_{11}(x,y)}{\partial y^2} s_{mx}^2 s_{ny}^2\Bigg]dxdy$$

$$C_{mn} = \int_0^{L_2}\int_0^{L_1}\mu(x,y)s_{mx}^2 s_{ny}^2\,dxdy \tag{62}$$

$$I_{mn} = \int_0^{L_2}\int_0^{L_1}\rho_0(x,y)s_{mx}^2 s_{ny}^2\,dxdy \tag{63}$$

Within the above expressions,

$$c_{mx} = \cos(m\pi x/L_1)\,, \qquad s_{mx} = \sin(m\pi x/L_1) \tag{64a,b}$$

$$c_{ny} = \cos(n\pi y/L_2)\,, \qquad s_{ny} = \sin(n\pi y/L_2) \tag{65a,b}$$

Eq. (60) can be normalized by dividing through by I_{mn} which results in,

$$\ddot{W}_{mn}(t) + 2\Delta_{mn}\,\omega_{mn}\dot{W}_{mn}(t) + \omega_{mn}^2 W_{mn}(t) = \tilde{P}_{mn}(t) \tag{66}$$

where $\omega_{mn} = \sqrt{K_{mn}/I_{mn}}$ is the natural frequency of the plate, $\Delta_{mn} = C_{mn}/2I_{mn}\,\omega_{mn}$ is the overall normalized damping coefficient, and $\tilde{P}_{mn}(t) = 16\,P_t(t)/mn\,\pi^4 I_{mn}$ is the normalized transverse pressure.

7. Results and Discussion

After a thorough literature search was conducted, it was found that there is none to very little work commenced on multidirectional functionally graded materials and or plates. This area of research is very sparse at least. A couple of papers were found that introduce the subject such as [1] Birman and Byrd and [2] Birman et al. As a result, nothing can be found in terms of the theoretical developments to validate the deformation of the plates. On the other hand, some validations can be made with the natural frequency of the multidirectional functionally graded plates where the theory is simplified down to simpler cases. For this reason, The governing differential equation, Eq. (66), which governs the structural dynamic response of multidirectional functionally graded plates is utilized to determine the natural frequency for the special case of a unidirectional functionally graded plate at either the fully metal and or fully ceramic compositions. For all cases of comparisons, Ti-6Al-4V/Aluminum Oxide was used.

In Table 1., A comparison is made between the present theory and those reported by [3] Befarani et al. and [6] Bishop, for a simply supported square isotropic plate, at various modes

where the frequency is non- dimensionalized. Very good agreement is seen. For this isotropic case, ($M = 0, N_1 = 0,\ N_2 = 0$). In table 2., comparisons are made for the natural frequency in Hertz of a unidirectional functionally graded plate. Comparisons are made for both the fully metal (Ti-6Al-4V) and fully ceramic (Aluminum Oxide) compositions for two modes of vibration between the present theory and those reported by [3]-[6]. These comparisons are presented for the case of fully metal ($M = 0, N_1 = 0,\ N_2 = 0$) and for the case of fully ceramic ($M = 1, N_1 = \infty,\ N_2 = \infty$). Very good agreement can be seen. The material properties utilized for Ti-6Al-4V/Aluminum Oxide are provided as,

Ti-6Al-4V: $E = 105.7 \times 10^9\ Pa, \quad \nu = 0.298, \quad \rho = 4429\ kg/m^3$

Aluminum Oxide: $E = 320.2 \times 10^9\ Pa, \quad \nu = 0.260, \quad \rho = 3750\ kg/m^3$

For the present case, various relationships between the geometrical and or material parameters and their effect on the structural response due to blast have been studied. Fig 2. Which compares the central deflection as a function of time for the isotropic cases (Ceramic and metal) as well as for 2 types of functionally graded materials (Bi-directional and multidirectional). It clearly shows that the metal composition has higher amplitudes and lower frequencies of oscillations in the absence of damping than for the cases. Ceramic appears to be the best performer from an amplitude standpoint but has the highest frequency of oscillation. The Multidirectional and Bidirectional case fall in between with multidirectional being the best performer between the two.

Fig 3. Depicts how various aspect ratios effects the deformation-time response of a multidirectional functionally graded plate (MDFGP). The optimum response appears to reside with an aspect ratio of 1. As the aspect ratios get smaller the deflections become larger with a

lower frequency. As the aspect ratios become larger the deflections again appear to be larger but smaller than the case of smaller aspect ratios. Also, the frequencies are lower.

In Fig 4. The effcet of various amounts of damping on the structural response provides results as you would expect. As the amount of damping is increased, the oscillations decay faster over time. For a fixed amount of damping, comparisons on the central deflection for the cases of fully metal, fully ceramic, bidirectional, and multidirectional are presented in Fig 5. It is apparent again that like the response in Fig 2. That ceramic is the best performer. With metal inherently being the worst. Again the bidirectional and multidirectional cases are in between with multidirectional superseding the bidirectional behavior from a performance standpoint in regards to the severity of the amplitude of deflection the frequency of oscillation. As previously seen metal has higher amplitudes with lower frequencies and vice versa for ceramic with all other cases in between. In the last figure, Fig 6., the effect of the amount of decay of the blast pressure has on a MDFGP is depicted. As the amount of decay is increased it shows that the response is diminished somewhat. To substantially diminish the amount of decay for a MDFGP, a much larger decay parameter would have to be used.

8. Conclusions

In conclusion, a linear theory of thin multidirectional functionally graded plates has been provided where the grading occurs in all three Orthogonal Cartesian Coordinate directions. Comparisons have been made between plates of other types of grading such as isotropic (metal and ceramic) and bidirectional. It has been found that ceramic is the best performer under blast. Although the frequency of oscillation is higher the deformation is greatly reduced. Ceramic due to its stiffness id very rigid. On the other hand depending on how brittle the ceramic is could also

play a factor. Within the functionally graded family, multidirectional functionally graded plates seem to perform the best as compared with bidirectional functionally graded plates. Further study should demand a determination of the stress distribution within these types of structures and material makeup to be the subject of further research. It is hoped that this current study will lay the ground work for more complex theoretical cases and other types of boundary conditions. Most likely all other cases will require advanced numerical techniques as part of the solution process.

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Figure Captions

Fig 1. A pictorial representation of a simply supported multidirectional functionally graded plate exposed to a spherical blast above the plate.

Fig 2. Comparison of the amplitude of center deflection as a function of time for various types of functionally graded plates.

Fig 3. The amplitude of deflection as a function of time for a multidirectional functionally graded plate for various aspect ratios.

Fig 4. The amplitude of deflection as a function of time for a multidirectional functionally graded plate for various amounts of damping.

Fig 5. Comparison of the amplitude of center deflection as a function of time for various types of functionally graded plates for a fixed amount of damping.

Fig 6. The amplitude of deflection as a function of time for a multidirectional functionally graded plate for various amounts of decay of the blast pressure.

Tables

Table 1. Comparison of the non-dimensional frequency $\varpi = \omega L^2 \sqrt{12\rho_0(1-\nu^2)/Eh^3}$ for simply supported boundary conditions of a square plate ($L_1 = L_2 = 0.4\ m,\ h = 0.002$ m)

ω_{mn}	**Present**	Reference[6]	Reference[3]
ω_{11}	**19.71**	19.74	19.76
ω_{12}	**49.27**	49.35	49.37

ω_{13}	**98.55**	98.70	98.74

Table 2. Comparison of the natural frequency ω (Hz) for a simply supported square Ti-6Al-4V/ Aluminum Oxide functionally graded plate ($L_1 = L_2 = 0.4\ m,\ h = 0.002$ m)

	Mode	**Present**	Reference[3]	Reference[5]	Reference[6]	Reference[4]
Fully Metal	m=1,n=1	**144.89**	143.4	143.67	145.04	144.66
	m=1,n=2	**362.23**	358.42	360.64	362.61	360.53
Fully Ceramic	m=1,n=1	**270.93**	273.906	268.60	271.23	268.92
	m=1,n=2	**677.33**	685.003	674.38	678.06	669.40

Figures (In numerical Order)

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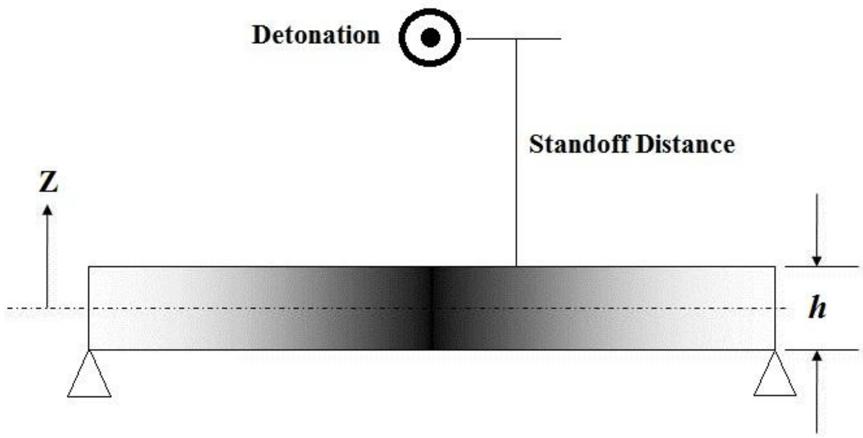
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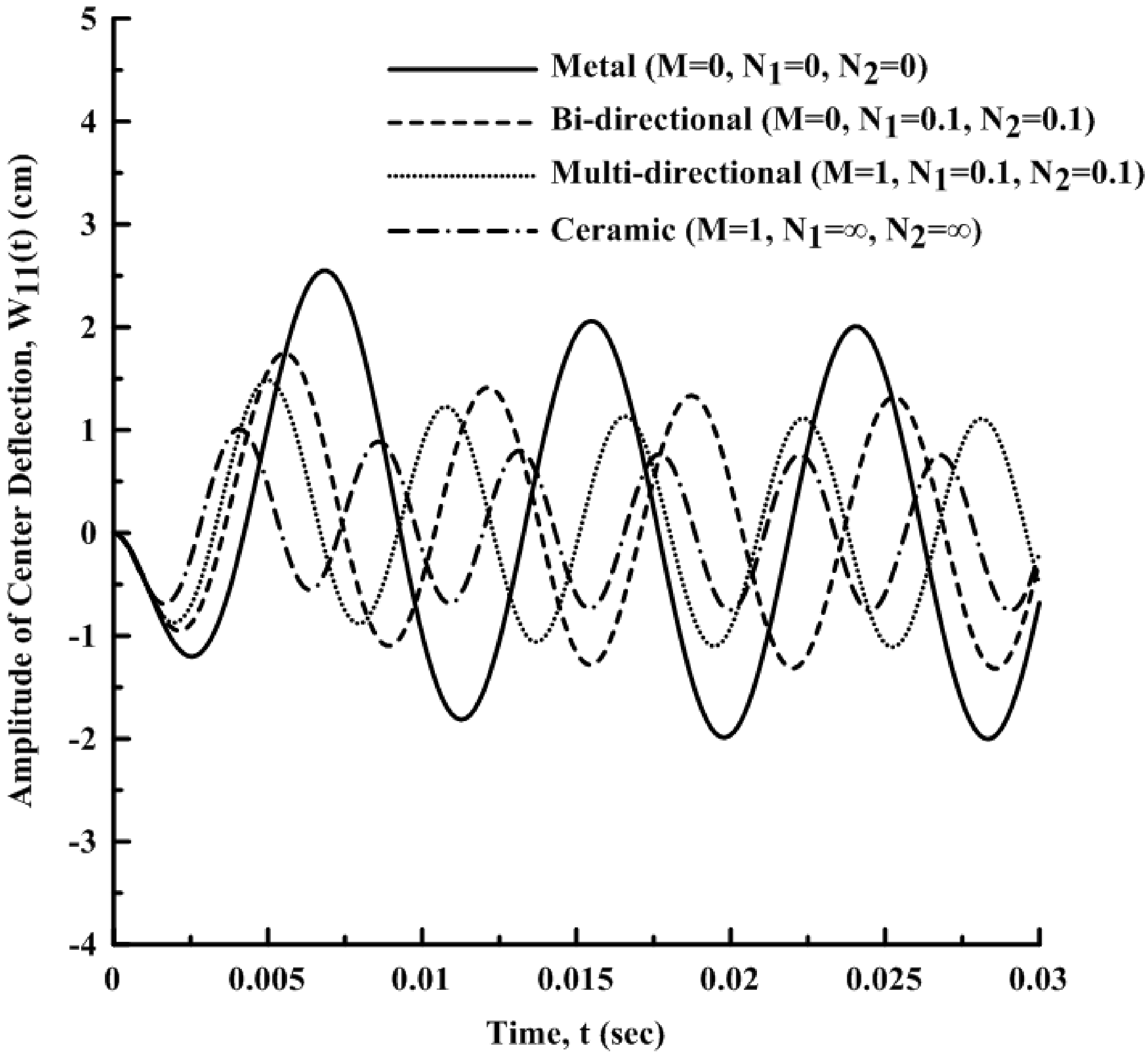
h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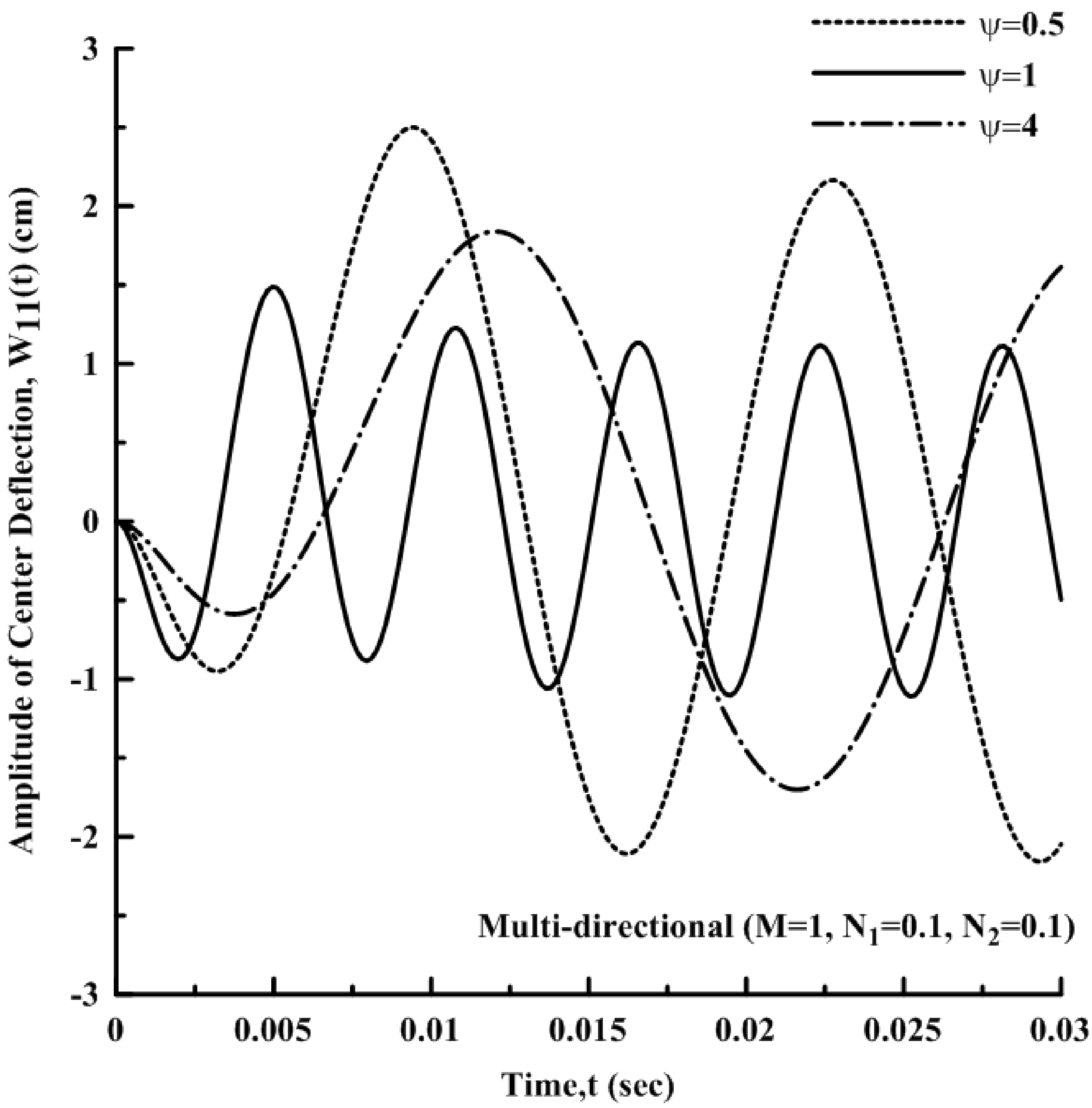
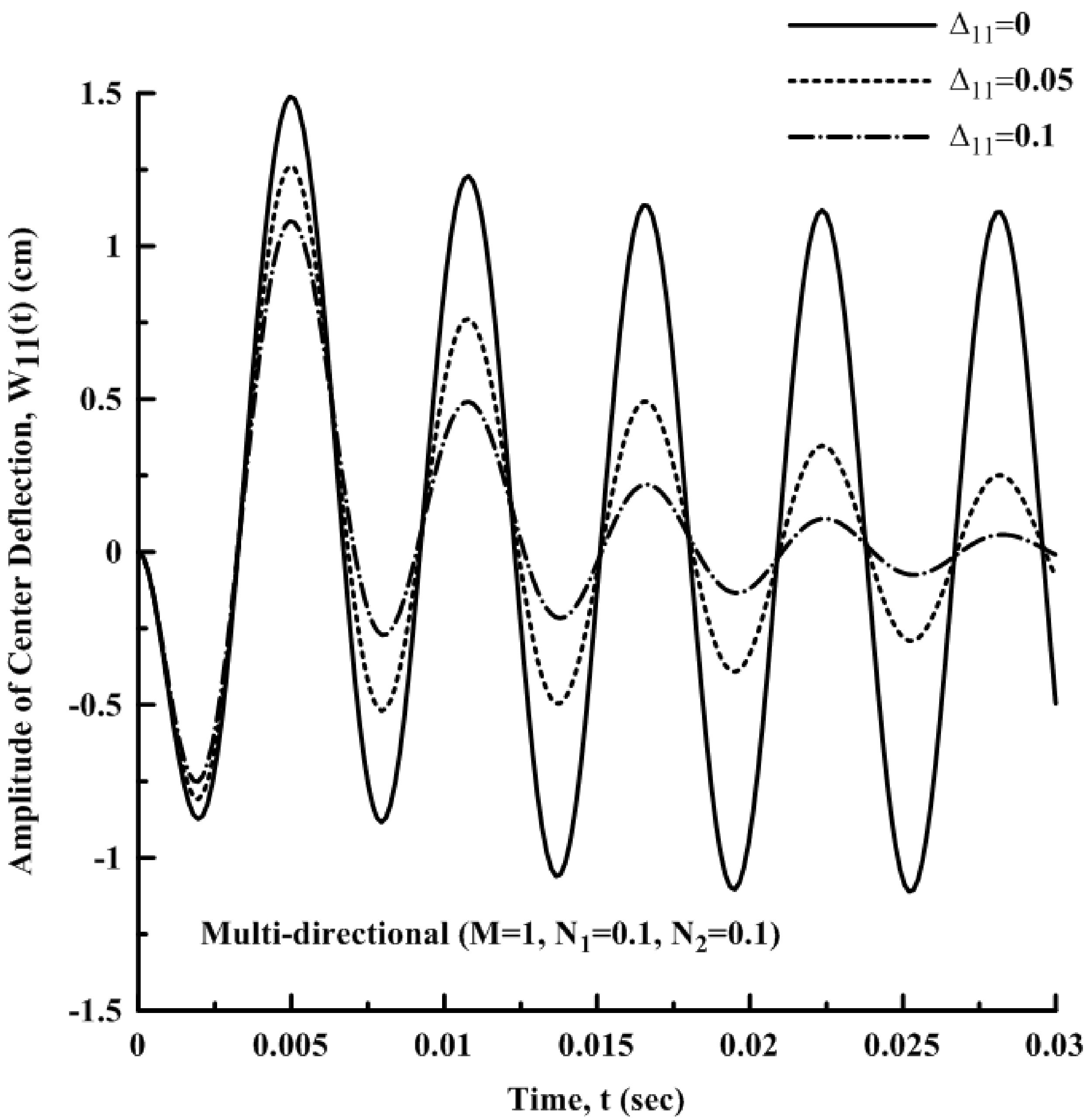
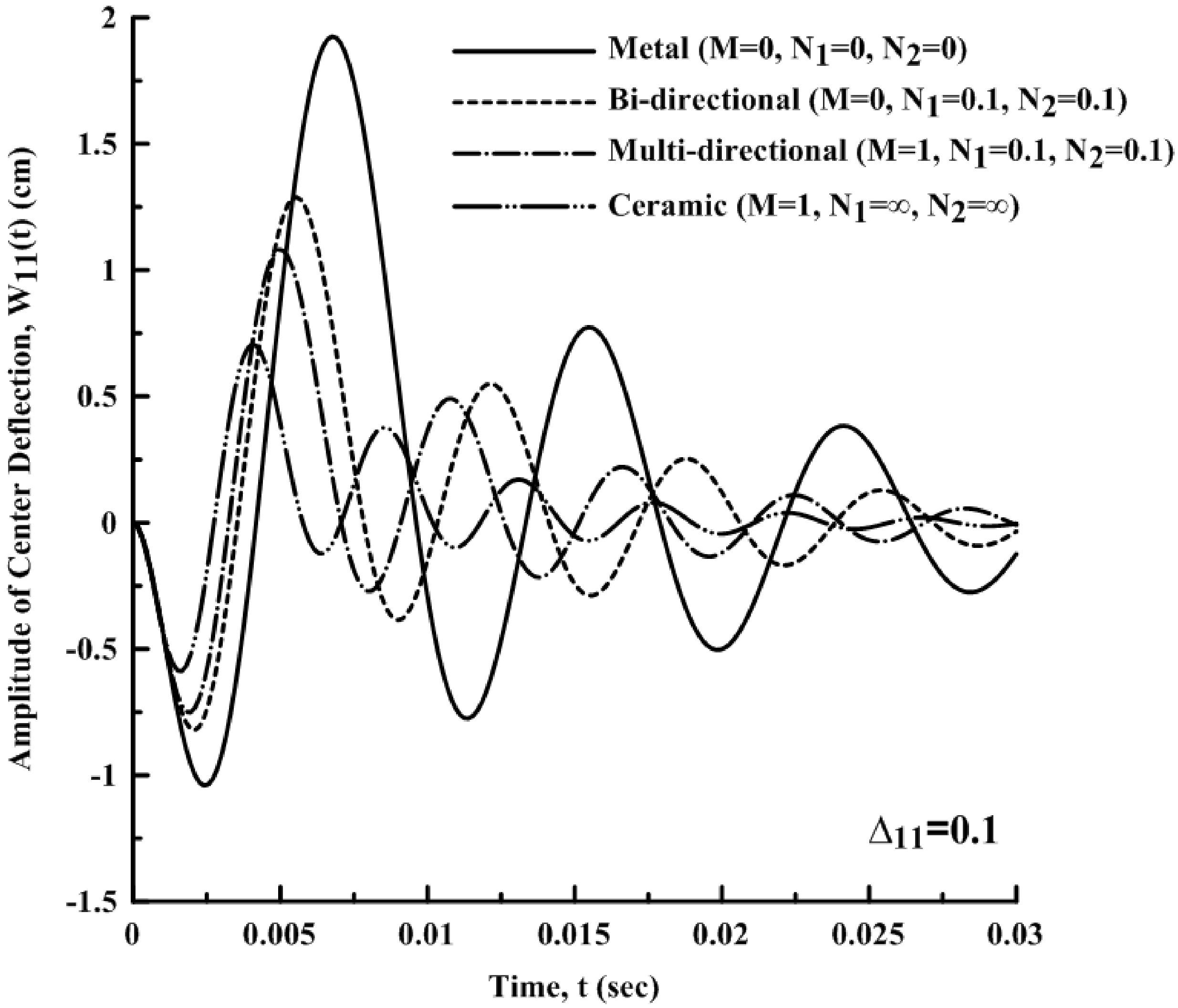
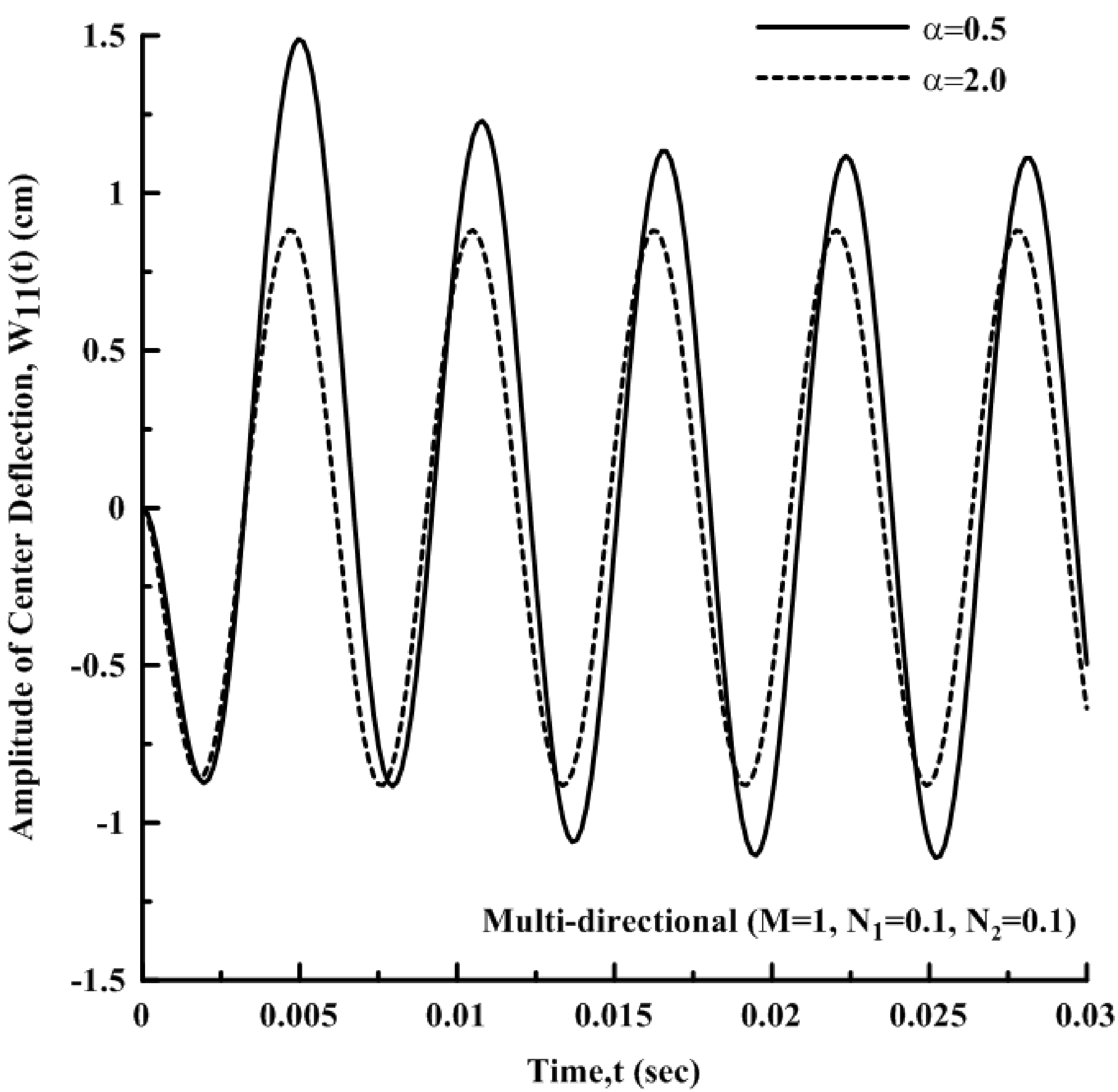
Bi-directional (M=0, N1=0.1, N2=0.1)
Multi-directional (M=1, N1=0.1, N2=0.1)
Ceramic (M=1, N1=∞, N2=∞)
Amplitude of Center Deflection, W11(t) (cm)
Time, t (sec)
5
4
3
2
1
0
-1
-2
-3
-4
0
0.005
0.01
0.015
0.02
0.025
0.03


ψ=0.5
ψ=1
ψ=4
Amplitude of Center Deflection, W11(t) (cm)
3
2
1
0
-1
-2
-3
Multi-directional (M=1, N1=0.1, N2=0.1)
0
0.005
0.01
0.015
0.02
0.025
0.03
Time,t (sec)


Δ11=0
Δ11=0.05
Δ11=0.1
1.5
1
0.5
0
-0.5
-1
-1.5
Amplitude of Center Deflection, W11(t) (cm)
Multi-directional (M=1, N1=0.1, N2=0.1)
0
0.005
0.01
0.015
0.02
0.025
0.03
Time, t (sec)


Metal (M=0, N1=0, N2=0)
Bi-directional (M=0, N1=0.1, N2=0.1)
Multi-directional (M=1, N1=0.1, N2=0.1)
Ceramic (M=1, N1=∞, N2=∞)
Amplitude of Center Deflection, W11(t) (cm)
Time, t (sec)
Δ11=0.1
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.005
0.01
0.015
0.02
0.025
0.03



α=0.5
α=2.0
1.5
1
0.5
0
-0.5
-1
-1.5
Amplitude of Center Deflection, W11(t) (cm)
Multi-directional (M=1, N1=0.1, N2=0.1)
0
0.005
0.01
0.015
0.02
0.025
0.03
Time,t (sec)